Pass back exams

# Back to Trig Substitution

We used the substitution

$u = \arcsin\left(\frac{x}{5}\right)$

(or $x = 5\sin(u)$)

to solve the integral

$$\int_0^1 \sqrt{25 - x^2}\, dx$$

We'll usually write $x = 5\sin\theta$

instead of $x = 5\sin(u)$

## Example 1

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

$$x = \tan(\theta)$$

$$(\tan^2\theta + 1 = \sec^2\theta)$$

$$dx = \sec^2(\theta)\, d\theta$$

Make the substitution.

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

$$= \int \frac{\sec^2(\theta)\, d\theta}{\tan^2(\theta)\sqrt{\underbrace{1+\tan^2(\theta)}_{=\sec^2\theta}}}$$

$$= \int \frac{\sec^2(\theta)\, d\theta}{\tan^2(\theta)\sqrt{\sec^2(\theta)}}$$

$$= \int \frac{\sec^{\cancel{2}}(\theta)\, d\theta}{\tan^2(\theta)\,\cancel{\sec(\theta)}}$$

$$= \int \frac{\sec(\theta)}{\tan^2(\theta)}\, d\theta$$

Convert to sines + cosines

$$= \int \frac{1}{\cos\theta} \cdot \cot^2\theta \, d\theta$$

$$= \int \frac{1}{\cancel{\cos(\theta)}} \cdot \frac{\cos^{\cancel{2}}(\theta)}{\sin^2(\theta)}\, d\theta$$

$$= \int \frac{\cos(\theta)}{\sin^2(\theta)}\, d\theta$$

$u = \sin(\theta)$

$du = \cos(\theta)\, d\theta$

We get

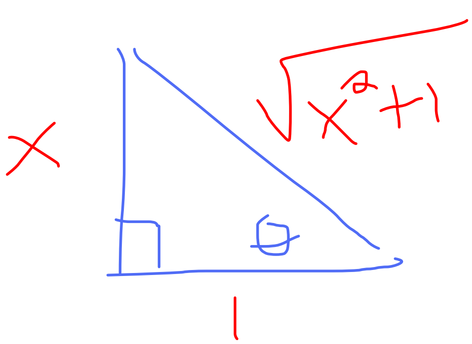
$$\int \frac{1}{u^2}\,du = -\frac{1}{u} + C$$

$\frac{1}{u^2} = u^{-2}$

$$= -\frac{1}{\sin(\theta)} + C$$

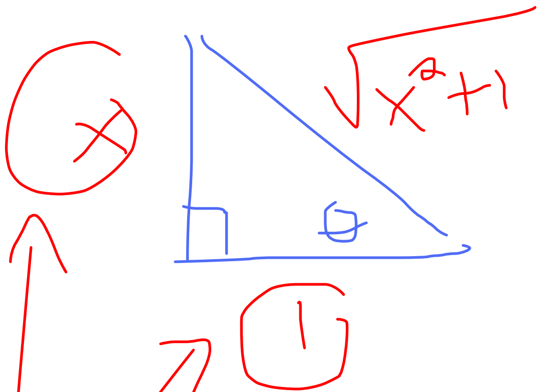
$x = \tan(\theta)$, so $\theta = \arctan(x)$

$$= -\frac{1}{\sin(\arctan(x))} + C$$




$$\sin(\theta) = \frac{opp}{hyp}$$

$$= \frac{x}{\sqrt{x^2+1}}$$




$$\sin(\theta) = \frac{opp}{hyp} = \frac{x}{\ }$$

Want to know $\sin(\underbrace{\arctan(x)}_{\theta})$

$$\tan\theta = \tan(\arctan(x)) = x = \frac{opp}{adj}$$

$$x = \frac{x}{1} \qquad \sin(\theta) = \frac{x}{\sqrt{x^2+1}}$$

Final answer:

$$\boxed{-\frac{\sqrt{x^2+1}}{x} + C}$$

Example 2 :

$$\int_{1}^{\ln(5)} \frac{e^{2x}}{\sqrt{e^{4x}-1}}\,dx$$

$$u = e^{2x} \quad \left(e^{4x}-1 = \left(e^{2x}\right)^2 - 1\right)$$

$$du = 2e^{2x}\,dx \rightarrow \frac{du}{2} = e^{2x}\,dx$$

$$u(1) = e^2$$

$$u(\ln(5)) = 25$$

Integral becomes $\frac{1}{2}\int_{e^2}^{25} \frac{du}{\sqrt{u^2-1}}$

$$\frac{1}{2}\int_{e^2}^{25} \frac{du}{\sqrt{u^2-1}}$$

$$u = \sec(\theta)$$

$$du = \sec(\theta)\tan(\theta)\,d\theta$$

forget bounds!

we get

$$\frac{1}{2}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sqrt{\underbrace{\sec^2(\theta)-1}_{\tan^2(\theta)}}}$$

We have

$$\frac{1}{2}\int \frac{\sec(\theta)\cancel{\tan(\theta)}}{\cancel{\tan(\theta)}}\,d\theta$$

$$= \frac{1}{2}\int \sec(\theta)\,d\theta$$

Done after break!

$$\boxed{\frac{1}{2}\left(\ln\left|25+\sqrt{624}\right| - \ln\left|e^{2}+\sqrt{e^{4}-1}\right|\right)}$$

## General (though not foolproof) Rules

$x^2 + a^2$ use $x = a\tan(\theta)$

$a^2 - x^2$ use $x = a\sin(\theta)$

$x^2 - a^2$ use $x = a\sec(\theta)$

# Partial Fractions

(Section 7.4)

The last integration trick for a while.

# Integrals of Trig Functions

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos(x)| + C$$

$$\int \cot(x)\,dx = \ln|\sin(x)| + C$$

Example 3:

$$\int \sec(x)\,dx$$

$$\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

No trick seems to work,
so convert to cosine

$$\int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx$$

$$= \int \frac{1}{\cos(x)} \cdot \underbrace{\frac{\cos(x)}{\cos(x)}}_{=1}\,dx$$

This equals $\int \frac{\cos(x)}{\cos^2(x)}\,dx$

and since $\cos^2(x) = 1 - \sin^2(x)$,

we get

$$\int \frac{\cos(x)}{1-\sin^2(x)}\,dx$$

$$u = \sin(x)$$

$$du = \cos(x)\,dx.$$

The integral becomes $\int \frac{du}{1-u^2}$

$$\int \frac{du}{1-u^2}$$

$$= \int \frac{1}{(1-u)(1+u)}\, du$$

I claim there are numbers $A$ and $B$ with

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

$A = B = \frac{1}{2}$ since

$$\frac{1}{2}\left(\frac{1}{1+v}+\frac{1}{1-v}\right)$$

$$=\frac{1}{2}\left(\frac{1-v}{(1+v)(1-v)}+\frac{1+v}{(1+v)(1-v)}\right)$$

$$=\frac{1}{2}\left(\frac{1-\cancel{v}+1+\cancel{v}}{(1+v)(1-v)}\right)$$

$$=\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{(1+v)(1-v)}\right)$$

$$=\frac{1}{(1+v)(1-v)}$$

This gives us that

$$\int \frac{1}{(1-u)(1+u)}\,du$$

$$= \frac{1}{2}\int\left(\frac{1}{1-u}+\frac{1}{1+u}\right)du$$

$$= \frac{1}{2}\left(\int \underbrace{\frac{1}{1-u}}\,du + \int \underbrace{\frac{1}{1+u}}\,du\right)$$

$w = 1-u$

$dw = -du$

$s = 1+u$

$ds = du$

$$= \frac{1}{2}\left(-\int \frac{1}{w}\,dw + \int \frac{1}{s}\,ds\right)$$

$$\frac{1}{2}\left(-\int \frac{1}{w}\,dw + \int \frac{1}{s}\,ds\right)$$

$$= \frac{1}{2}\left(-\ln|w| + \ln|s|\right) + C$$

$$= \frac{1}{2}\left(\ln\left(\frac{|s|}{|w|}\right)\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{|1+u|}{|1-u|}\right) + C$$

remember $u = \sin(x)$

$$= \boxed{\frac{1}{2}\ln\left(\frac{|1+\sin(x)|}{|1-\sin(x)|}\right) + C}$$

The book will say

$$\int \sec(x)\,dx = \ln\left|\sec(x) + \tan(x)\right| + C$$

2 points extra credit

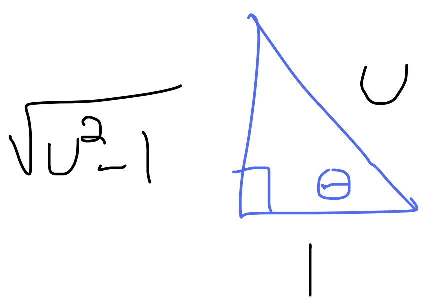
Show that these are equivalent!

Due next Tuesday

# Back to Example 2.

$$\frac{1}{2}\int \sec(\theta)\,d\theta = \frac{1}{2}\ln\left|\sec(\theta)+\tan(\theta)\right|$$

$$u = \sec(\theta) = \frac{hyp}{adj}$$




So $\tan(\theta) = \sqrt{u^2-1}$,

and we get

$$\frac{1}{2}\ln\left|u + \sqrt{u^2-1}\right|$$

bounds $e^2$ to $25$

$$\frac{1}{2}\ln\left|u+\sqrt{u^2-1}\right|\Big|_{e^2}^{25}$$

$$=\boxed{\frac{1}{2}\left(\ln\left|25+\sqrt{624}\right|-\ln\left|e^2+\sqrt{e^4-1}\right|\right)}$$

Why care?

<u>Example 4:</u> $\displaystyle\int_{1}^{2/\sqrt{3}} \sqrt{x^2-1}\, dx$

$x = \sec(\theta)$

$dx = \tan(\theta)\sec(\theta)\,d\theta$

putting all this in:

$$\int_{0}^{\frac{\pi}{6}} \underbrace{\tan^2(\theta)}_{\sec^2(\theta)-1} \sec(\theta)\, d\theta$$

$$\int_0^{\frac{\pi}{6}} \left(-\sec\theta + \sec^3(\theta)\right) d\theta$$

$$= -\underbrace{\int_0^{\frac{\pi}{6}} \sec(\theta)\, d\theta}_{\shortparallel} + \underbrace{\int_0^{\frac{\pi}{6}} \sec^3(\theta)\, d\theta}_{?}$$

$$\ln\left|\sec(\theta) + \tan(\theta)\right| \Big|_0^{\frac{\pi}{6}}$$

$$\int \sec^3(\theta)\, d\theta = \int \sec(\theta) \sec^2(\theta)\, d\theta$$

$u = \sec(\theta)$ $\quad$ $v = \tan(\theta)$

$du = \sec(\theta)\tan(\theta)\, d\theta$ $\quad$ $dv = \sec^2(\theta)$

So $\boxed{\int \sec^3(\theta)\,d\theta}$

$= \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)\,d\theta$

$= \sec(\theta)\tan(\theta) - \int \sec(\theta)\left(\sec^2(\theta) - 1\right)d\theta$

$= \sec(\theta)\tan(\theta) - \boxed{\int \sec^3(\theta)\,d\theta} + \int \sec(\theta)\,d\theta$

add $\int \sec^3(\theta)\,d\theta$ to both sides

$2\int \sec^3(\theta)\,d\theta = \sec(\theta)\tan(\theta) + \int \sec(\theta)\,d\theta$

So $\boxed{\int \sec^3(\theta)\,d\theta = \frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln\left|\sec\theta + \tan(\theta)\right|\right)}$

Finally,

$$\int_0^{\pi/6} \left(-\sec(\theta) + \sec^3(\theta)\right) d\theta$$

$$= \left(-\ln\left|\sec(\theta)+\tan(\theta)\right| \Big|_0^{\frac{\pi}{6}}\right)$$

$$+ \frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln\left|\sec(\theta)+\tan(\theta)\right|\right)\Big|_0^{\frac{\pi}{6}}$$

$$= -\frac{1}{2}\left(\ln\left|\sec(\theta)+\tan(\theta)\right| - \sec(\theta)\tan(\theta)\right)\Big|_0^{\frac{\pi}{6}}$$

$$= \boxed{\frac{-1}{2}\left(\ln(\sqrt{3}) - \frac{2}{3}\right)}$$